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Journal of Approximation Theory 128 (2004) 207–233
Journal of Approximation Theory
http://www.elsevier.com/locate/jat

Kolmogorov numbers of Riemann–Liouville operators over small sets and applications to Gaussian processes

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Received 13 August 2003; accepted in revised form 27 April 2004

Communicated by Peter Oswald

Abstract

We investigate compactness properties of the Riemann–Liouville operator R_α of fractional integration when regarded as operator from $L_2[0,1]$ into $C(K)$, the space of continuous functions over a compact subset K in $[0,1]$. Of special interest are small sets K, i.e. those possessing Lebesgue measure zero (e.g. fractal sets). We prove upper estimates for the Kolmogorov numbers of R_α against certain entropy numbers of K. Under some regularity assumption about the entropy of K these estimates turn out to be two-sided. By standard methods the results are also valid for the (dyadic) entropy numbers of R_α. Finally, we apply these estimates for the investigation of the small ball behavior of certain Gaussian stochastic processes, as e.g. fractional Brownian motion or Riemann–Liouville processes, indexed by small (fractal) sets.

MSC: 47B06; 26A33; 60G15; 28A80

Keywords: Fractional integration; Kolmogorov numbers; Entropy numbers; Fractal sets; Small deviation; Fractional Brownian motion

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2004.04.011

1. Introduction

The aim of the present paper is to investigate compactness properties of the Riemann–Liouville fractional integration operator R_α when regarded as an operator from $L_2[0,1]$ into $C(K)$ (the space of continuous functions over K) for certain compact subsets $K \subseteq [0,1]$. Here, as usual, the operator R_α is defined by

$$(R_\alpha f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t \in [0,1]. \tag{1.1}$$

Properties of R_α as operator from $L_2[0,1]$ into $C(K)$ are of special interest for "small" sets K, i.e. those with Lebesgue measure zero. To our opinion those questions are interesting in their own right, although our main motivation for their investigation came from the theory of stochastic processes. Recall that R_α is tightly related to the fractional Brownian motion B_H of Hurst index $H = \alpha - \frac{1}{2}$ as well as to the so-called Riemann–Liouville process W_H (cf. [13,17,18] or Section 6). Thus our results lead to a deeper insight into the structure of B_H and W_H when indexed by "small" subsets K in $[0,1]$ (e.g. fractal sets). From a probabilistic point of view similar questions were recently treated in [19] and led to new properties for a large class of Lévy processes. Let us also mention some related results in [3] where the authors investigate compactness properties of integral operators in dependence of the entropy numbers of the underlying compact set.

We shall use two different quantities to measure the degree of compactness of R_α, namely Kolmogorov and (dyadic) entropy numbers. Let us shortly recall their definition.

If S is a compact operator from a Banach space E into a Banach space F its Kolmogorov numbers $d_n(S)$ are defined by

$$d_n(S) = d_n(S : E \to F) := \inf \left\{ \sup_{\|x\|_E \leqslant 1} d_F(Sx, F_n) \colon F_n \subseteq F,\ \dim(F_n) < n \right\}, \tag{1.2}$$

where, as usual,

$$d_F(y, F_n) := \inf\{ \|y - z\|_F \colon z \in F_n \}$$

denotes the distance of $y \in F$ to the subspace F_n (w.r.t. the norm in F).

The (dyadic) entropy numbers of S are given by

$$e_n(S) = e_n(S : E \to F) := \inf \left\{ \varepsilon > 0 : S(B_E) \subset \bigcup_{j=1}^{2^{n-1}} (y_j + \varepsilon B_F) \right\}.$$

Here B_E and B_F denote the (closed) unit balls in E and F, respectively. In other words, $e_n(S)$ is the infimum over all $\varepsilon > 0$ such that $S(B_E)$ can be covered by at most 2^{n-1} balls of radius $\varepsilon > 0$ in F. We refer to [6,24–26] for more information about Kolmogorov and entropy numbers.

As shown in [5,7,23] these two numbers are tightly related. For example, if an operator S maps a Hilbert space $\mathcal{H}$ into a Banach space E, then it holds

$$d_n(S:\mathcal{H}\to E)\approx n^{-\gamma}\cdot(\log n)^{\beta}$$

for some $\gamma>\frac{1}{2}$ and $\beta\in\mathbb{R}$ iff

$$e_n(S:\mathcal{H}\to E)\approx n^{-\gamma}\cdot(\log n)^{\beta}.$$

Here we have used the following notation. Given two sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ of positive real numbers we write $a_n\preccurlyeq b_n$ provided that $a_n\leqslant c\cdot b_n$ for a certain $c>0$. If, furthermore, also $b_n\preccurlyeq a_n$, then we write $a_n\approx b_n$.

Let us come back to R_α as defined in (1.1). First note that R_α maps $L_2[0,1]$ into $L_q[0,1]$ for a certain $q\geqslant 1$ iff $\alpha>\max\{0,1/2-1/q\}$. Moreover, if $2\leqslant q\leqslant\infty$, then

$$d_n(R_\alpha:L_2[0,1]\to L_q[0,1])\approx e_n(R_\alpha:L_2[0,1]\to L_q[0,1])\approx n^{-\alpha} \tag{1.3}$$

(cf. [1,11,17]). Observe that for $\alpha>\frac{1}{2}$ the functions $R_\alpha f$, $f\in L_2[0,1]$, are continuous, thus in this case we may regard R_α as operator from $L_2[0,1]$ into $C[0,1]$. Of course, the asymptotic in (1.3) remains valid in this case as well.

Given a compact subset $K\subseteq[0,1]$, then for $\alpha>\frac{1}{2}$ the operator R_α may be regarded in natural way as operator from $L_2[0,1]$ into $C(K)$, i.e. we investigate $R_\alpha f$ with respect to the norm

$$||R_\alpha f||_{C(K)}=\sup_{t\in K}|(R_\alpha f)(t)|,\quad f\in L_2[0,1].$$

Intuitively it is clear that the degree of compactness of R_α should increase (i.e. its Kolmogorov and/or entropy numbers tend to zero faster) provided that K becomes smaller. To make this more precise we need some suitable measure for the size of the compact set K. At a first glance one might expect the Hausdorff dimension of K as such a measure. Yet it turns out that this not the right quantity for our purposes. More suited are quantities related with the so-called box dimension of K (cf. [9]), i.e. we describe the size of K by its covering properties. More precisely, an adequate tool for the size of K is the behavior of its entropy numbers $\varepsilon_m(K)$ defined by

$$\varepsilon_m(K):=\inf\left\{\delta>0:K\subseteq\bigcup_{j=1}^{m}\Delta_j,\ \Delta_j\text{ intervals of length}<\delta\right\}. \tag{1.4}$$

If $\frac{1}{2}<\alpha\leqslant\frac{3}{2}$, then R_α is known to map $L_2[0,1]$ into $C^{\alpha-1/2}[0,1]$, the space of $(\alpha-\frac{1}{2})$–Hölder continuous functions over $[0,1]$. Hence, for those α's quite general assertions about so-called Hölder operators apply and the results in [4,6,30] lead to the following:

Proposition 1.1. *Suppose* $\frac{1}{2}<\alpha\leqslant\frac{3}{2}$ *and* $\varepsilon_m(K)\leqslant h(m)$ *for a regularly varying decreasing function* h. *Then this implies*

$$d_m(R_\alpha:L_2[0,1]\to C(K))\leqslant c\cdot m^{-1/2}\cdot h(m)^{\alpha-1/2}. \tag{1.5}$$

The disadvantage of the preceding result is that it does not apply for large α's. This is somehow surprising because the larger α the smoother the functions $R_\alpha f$ are. Thus our main goal was to extend Proposition 1.1 to arbitrary $\alpha > \frac{1}{2}$ and, moreover, to estimate the Kolmogorov numbers of R_α directly by the entropy numbers of the underlying set K. In the latter problem we did not succeed completely, because the case $\frac{1}{2} < \alpha < 1$ is not covered by the following main result of this paper.

Theorem 1.2. *Let $\alpha \geqslant 1$. Then there is a $\kappa \in \mathbb{N}$ such that for all compact sets $K \subseteq [0,1]$ it follows that*

$$d_{\kappa m}(R_\alpha : L_2[0,1] \to C(K)) \leqslant c \cdot m^{-1/2} \cdot \varepsilon_m(K)^{\alpha - 1/2}, \quad m \in \mathbb{N}. \tag{1.6}$$

For example, one may choose $\kappa = 4\,[\alpha] + 11$.

Applying Carl's inequality (cf. [5]) to Theorem 1.2 we get the following estimate for the entropy numbers of the Riemann–Liouville integration operator.

Theorem 1.3. *Let $K \subseteq [0,1]$ be compact and suppose that $\varepsilon_m(K) \leqslant h(m)$, $m \in \mathbb{N}$, for some decreasing function h satisfying $\sup_{m \geqslant 1} h(m)/h(2m) := \lambda < \infty$. Then for $\alpha \geqslant 1$ there is a $c > 0$ (only depending on λ and α) such that*

$$e_m(R_\alpha : L_2[0,1] \to C(K)) \leqslant c \cdot m^{-1/2} \cdot h(m)^{\alpha - 1/2}. \tag{1.7}$$

Remark. Estimate (1.7) is tightly related to results presented in [33]. But observe that there only sets K are investigated which satisfy some kind of self-similarity while our estimates apply to arbitrary compact subsets of the unit interval.

Furthermore, we prove that under some regularity assumptions about $\varepsilon_m(K)$ estimate (1.6) is optimal. More precisely, the following will be shown.

Theorem 1.4. *Suppose $\alpha > \frac{1}{2}$ and let $K \subseteq [0,1]$ be a compact set such that for some $\lambda \geqslant 1$ we have*

$$\varepsilon_m(K) \leqslant \lambda \cdot \varepsilon_{2m}(K), \qquad m \in \mathbb{N}. \tag{1.8}$$

Then it follows that

$$d_m(R_\alpha : L_2[0,1] \to C(K)) \geqslant c \cdot m^{-1/2} \cdot \varepsilon_m(K)^{\alpha - 1/2}, \tag{1.9}$$

where $c > 0$ only depends on α and λ.

Combining the preceding Theorem with Theorem 1.2 (resp. Proposition 1.1 for $\frac{1}{2} < \alpha < 1$) and with the above-mentioned relation between the entropy and Kolmogorov numbers for operators defined on Hilbert spaces leads to the following.

Corollary 1.5. *Suppose that* $\varepsilon_m(K) \approx m^{-\theta} \cdot (\log m)^\beta$ *for some* $\theta \geqslant 1$ *and* $\beta \in \mathbb{R}$ (*note that by* $K \subseteq [0,1]$ *we necessarily have* $\beta \leqslant 0$ *for* $\theta = 1$). *Then this implies*

$$d_m(R_\alpha : L_2[0,1] \to C(K)) \approx e_m(R_\alpha : L_2[0,1] \to C(K)) \approx m^{-1/2-\theta(\alpha-1/2)} \cdot (\log m)^{\beta(\alpha-1/2)}.$$

The organization of the paper is as follows. In Section 2 we prove Theorem 1.2 first for integer α's. Then a multiplication formula (Lemma 3.2) allows us to deduce the general case from that of integer α's. This will be carried out in Section 3. In Section 4 we prove Theorem 1.4. Again we derive the proof from that for integer α's. Compact sets $K \subset [0,1]$ with Lebesgue measure zero admit a very special (Cantor like) representation which allows quite precise estimates for $\varepsilon_m(K)$. The representation as well as the two-sided entropy estimates will be subject of Section 5. Finally, in Section 6 some probabilistic applications will be stated and proved. For example, we determine the small ball behavior of fractional Brownian motions and Riemann–Liouville processes indexed by compact subsets K of $[0,1]$ in dependence of the size of K.

Throughout the paper c with or without subscript always denotes a positive constant (maybe different at each occurrence) which is either universal or depends only on the order of the Riemann–Liouville operator.

2. Proof of Theorem 1.2 for integer α's

As already mentioned, when $\alpha > \frac{3}{2}$ the results about Hölder continuous operators do no longer apply. Hence some completely different approach is necessary. The basic idea is to cover K in an optimal way by m intervals $\Delta_1, \ldots, \Delta_m$ with $|\Delta_j| < \delta$ and to prove very precise estimates (in dependence of δ and m) for $d_n(R_\alpha)$ as operator with values in $C(\Delta)$ where $\Delta := \bigcup_{j=1}^m \Delta_j$.

Let us fix the notation. Here and later on $\Delta_1, \ldots, \Delta_m$ are always intervals in $[0,1]$ with disjoint interior, say $\Delta_j = [a_j, b_j]$, $1 \leqslant j \leqslant m$, and $\Delta := \bigcup_{j=1}^m \Delta_j$. We may regard now R_α as operator from $L_2[0,1]$ into $C(\Delta)$ in the usual way, i.e.

$$(R_\alpha f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t \in \Delta. \tag{2.1}$$

When splitting R_α into m independent pieces we obtain an operator R_α^Δ mapping $L_2(\Delta)$ into $C(\Delta)$ acting as follows:

$$(R_\alpha^\Delta f)(t) := \frac{1}{\Gamma(\alpha)} \sum_{j=1}^m \int_{a_j}^t (t-s)^{\alpha-1} f(s)\, ds \cdot \mathbf{1}_{\Delta_j}(t), \quad t \in \Delta. \tag{2.2}$$

Our strategy is to compare the compactness properties of R_α with those of R_α^Δ in dependence of m and the length of the intervals. To this end we introduce operators

S_α^j, $1 \leqslant j \leqslant m$, mapping $L_2[0, a_j]$ into $C(\Delta_j)$ by

$$(S_\alpha^j f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^{a_j} (t-s)^{\alpha-1} f(s)\, ds, \quad t \in \Delta_j. \tag{2.3}$$

Since

$$R_\alpha - R_\alpha^\Delta = \sum_{j=1}^m S_\alpha^j \tag{2.4}$$

it is necessary to investigate properties of the S_α^j's more thoroughly.

Let $m = 1$, i.e. we have only one interval $\Delta = [a, b]$ and only one operator S_α defined by (2.3), i.e.

$$(S_\alpha f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^{a} (t-s)^{\alpha-1} f(s)\, ds, \quad a \leqslant t \leqslant b. \tag{2.5}$$

For each $\alpha > 0$ this operator maps $L_2[0, a]$ into $L_2(\Delta)$ and if $\alpha > \frac{1}{2}$, then S_α is even an operator into $C(\Delta)$.

A first result describes the structure of S_α for integer α's.

Lemma 2.1. *If α is an integer, then it follows that* $\mathrm{rk}(S_\alpha) \leqslant \alpha$.

Proof. Writing S_α as

$$(S_\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\alpha-1} \binom{\alpha-1}{k} \int_0^a (-s)^{\alpha-1-k} f(s)\, ds \cdot t^k \cdot \mathbf{1}_\Delta(t)$$

immediately proves the lemma. □

We are now in the position to estimate $d_n(R_\alpha : L_2[0,1] \to C(\Delta))$ in the case of integer α's.

Proposition 2.2. *Suppose $\alpha \in \mathbb{N}$ and let $\Delta_1, \ldots, \Delta_m$ be intervals in $[0, 1]$ as before with union Δ. Then for any $n \in \mathbb{N}$ we have*

$$d_{n+2m\alpha}(R_\alpha : L_2[0,1] \to C(\Delta)) \leqslant c \cdot n^{-\alpha} \cdot |\Delta|^{\alpha-1/2}. \tag{2.6}$$

In particular, if $|\Delta_j| < \delta$, $1 \leqslant j \leqslant m$, then it follows that

$$d_{(2\alpha+1)m}(R_\alpha : L_2[0,1] \to C(\Delta)) \leqslant c \cdot m^{-1/2} \cdot \delta^{\alpha-1/2}. \tag{2.7}$$

Proof. Since (2.4) and Lemma 2.1 imply for integer α's that $\mathrm{rk}(R_\alpha - R_\alpha^\Delta) \leqslant m\alpha$, we conclude $d_{m\alpha+1}(R_\alpha - R_\alpha^\Delta) = 0$. Using additivity properties of the d_n's this leads to

$$d_{n+m\alpha}(R_\alpha) \leqslant d_n(R_\alpha^\Delta) \quad \text{as well,} \tag{2.8}$$

$$d_{n+m\alpha}(R_\alpha^\Delta) \leqslant d_n(R_\alpha). \tag{2.9}$$

Both estimates are valid for any choice of m disjoint intervals $\Delta_1, \ldots, \Delta_m$ in $[0, 1]$. In particular, the remain true when we shift $\Delta_1, \ldots, \Delta_m$ to the left, i.e. when passing to

$\tilde{\Delta}_1, \dots, \tilde{\Delta}_m$ with $|\Delta_j| = |\tilde{\Delta}_j|$, $1 \leqslant j \leqslant m$, and

$$\tilde{\Delta} := \tilde{\Delta}_1 \cup \dots \cup \tilde{\Delta}_m = [0, |\Delta|].$$

We are going to apply (2.8) for $\Delta_1, \dots, \Delta_m$ and (2.9) for $\tilde{\Delta}_1, \dots, \tilde{\Delta}_m$. In the latter case the operator R_α (which we denote by $\tilde{R}_\alpha$ in order to distinguish it from the operator given by (2.1)) maps $L_2[0, |\Delta|]$ into $C[0, |\Delta|]$, hence by the scaling properties of R_α and by (1.3) we obtain

$$d_n(\tilde{R}_\alpha) \leqslant c \cdot |\Delta|^{\alpha - 1/2} \cdot n^{-\alpha}, \quad n \in \mathbb{N}. \tag{2.10}$$

Since R_α^Δ may be isometrically transformed into $R_\alpha^{\tilde{\Delta}}$ it follows that

$$d_n(R_\alpha^\Delta) = d_n(R_\alpha^{\tilde{\Delta}}), \quad n \in \mathbb{N}. \tag{2.11}$$

Hence by (2.8)–(2.11) we finally arrive at

$$\begin{aligned} d_{2\alpha m + n}(R_\alpha : L_2[0,1] \to C(\Delta)) &\leqslant d_{\alpha m + n}(R_\alpha^\Delta) = d_{\alpha m + n}(R_\alpha^{\tilde{\Delta}}) \\ &\leqslant d_n(\tilde{R}_\alpha) \leqslant c \cdot |\Delta|^{\alpha - 1/2} \cdot n^{-\alpha} \end{aligned}$$

as claimed.

Estimate (2.7) may be immediately derived from (2.6) by choosing $n = m$.

As consequence of Proposition 2.2 we may now prove Theorem 1.2 for special α's.

Proof of Theorem 1.2. *for integer α's*: Given a natural number m we choose a covering of K by m intervals $\Delta_1, \dots, \Delta_m$ such that $\delta := \sup_{1 \leqslant j \leqslant m} |\Delta_j| \leqslant 2 \cdot \varepsilon_m(K)$. Let as before Δ be the union of the Δ_j's. Then we define an operator $\Phi : C(\Delta) \to C(K)$ by $\Phi(f) := f|_K$ and obtain

$$[R_\alpha : L_2[0,1] \to C(K)] = \Phi \circ [R_\alpha : L_2[0,1] \to C(\Delta)].$$

Consequently, if $\alpha \in \mathbb{N}$, by Proposition 2.2 it follows that

$$\begin{aligned} d_{(2\alpha+1)m}(R_\alpha : L_2[0,1] \to C(K)) &\leqslant ||\Phi|| \cdot d_{(2\alpha+1)m}(R_\alpha : L_2[0,1] \to C(\Delta)) \\ &\leqslant c \cdot m^{-1/2} \cdot \delta^{\alpha - 1/2} \leqslant c' \cdot m^{-1/2} \varepsilon_m(K)^{\alpha - 1/2}. \end{aligned}$$

This completes the proof of (1.6) with $\kappa = 2\alpha + 1$. $\square$

3. Proof of Theorem 1.2—general case

We turn now to the case of non-integer α's. Here (2.8) and (2.9) are no longer valid, thus we have to find some substitute for these estimates.

We start with introducing another helpful sequence of so-called operator numbers. Let S be an operator from a separable Hilbert space $\mathcal{H}$ into a Banach space E. Then S is said to be an l-operator provided that

$$X_S := \sum_{k=1}^{\infty} \xi_k S f_k \tag{3.1}$$

converges a.s. in E for some (each) ONB $(f_k)_{k\geqslant 1}$ in $\mathcal{H}$. Here $(\xi_k)_{k\geqslant 1}$ denotes an i.i.d. sequence of $\mathcal{N}(0,1)$-distributed random variables. Whenever S is an l-operator, its l-norm is defined by

$$l(S) := (\mathbb{E}||X_S||^2)^{1/2}.$$

Given an l-operator $S:\mathcal{H}\to E$ we set

$$l_n(S) := \inf\{l(S-A):\ A:\mathcal{H}\to E,\ \mathrm{rk}(A)<n\}.$$

For properties of these numbers we refer to [13,17,26].

Let now S be a compact operator between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then S admits a so-called Schmidt representation, i.e.

$$Sh = \sum_{n=1}^{\infty} \sigma_n \langle h, f_n \rangle g_n$$

with $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant 0$ tending to zero and two orthonormal systems $(f_k)_{k\geqslant 1}$ and $(g_k)_{k\geqslant 1}$ in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The σ_n's are usually called the singular numbers of S. It is known (cf. [24, 11.3.3]) that then $d_n(S) = \sigma_n$ for $n\in\mathbb{N}$. Furthermore, S is an l-operator iff it is Hilbert–Schmidt, i.e. iff the σ_n's are square summable and, moreover, as easily can be seen (cf. [13]) then we have $l_n(S) = (\sum_{k=n}^{\infty} \sigma_k^2)^{1/2}$. In particular, it holds

$$\sqrt{n} d_{2n-1}(S) \leqslant l_n(S). \tag{3.2}$$

It is worthwhile to mention that a deep result due to Pajor and Tomczak–Jaegermann (cf. [22]) asserts that (3.2) remains valid (with some universal constant on the right-hand side) for l-operators with values in Banach spaces.

The following Lemma is crucial to get rid of a factor $\sqrt{m}$ later on. Before formulating it let us fix the notation. Given Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_m$ the Hilbert space $l_2(\mathcal{H}_j)$ is then defined by

$$l_2(\mathcal{H}_j) := \{x = (x_1, \dots, x_m):\ x_j \in \mathcal{H}_j\}$$

with norm $||x||_{l_2(\mathcal{H}_j)} := (\sum_{j=1}^{m} ||x_j||^2)^{1/2}$.

Lemma 3.1. *Let $S_1, \dots, S_m$ be l-operators mapping $\mathcal{H}$ into some Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_m$. Define $S:\mathcal{H}\to l_2(\mathcal{H}_j)$ by $Sh := (S_1 h, \dots, S_m h)$ for $h\in\mathcal{H}$. Then for each $n\in\mathbb{N}$ it follows that*

$$\sqrt{nm}\cdot d_{2nm-1}(S) \leqslant \left[\sum_{j=1}^{m} l_n(S_j)^2\right]^{1/2}. \tag{3.3}$$

Proof. Let $A_j:\mathcal{H}\to\mathcal{H}_j$ be operators of rank $<n$ such that

$$l(S_j - A_j) \leqslant (1+\varepsilon)\, l_n(S_j), \quad 1 \leqslant j \leqslant m,$$

for a given $\varepsilon > 0$. Define now $A : \mathcal{H} \to l_2(\mathcal{H}_j)$ by $Ah := (A_1 h, \ldots, A_m h)$ for $h \in \mathcal{H}$. Then we have $\mathrm{rk}(A) < nm$ and in view of

$$||(S - A)h||^2 = \sum_{j=1}^{m} ||(S_j - A_j)h||^2_{\mathcal{H}_j}$$

one easily gets

$$l_{mn}(S)^2 \leqslant l(S - A)^2 \leqslant \sum_{j=1}^{m} l(S_j - A_j)^2 \leqslant (1 + \varepsilon)^2 \sum_{j=1}^{m} l_n(S_j)^2. \tag{3.4}$$

Thus (3.3) follows directly from (3.2) and (3.4). □

Our next objective is to estimate the degree of compactness of $R_\alpha - R_\alpha^\Delta$ as defined in (2.1) and (2.2) in the case $\alpha \notin \mathbb{N}$. The basic idea is to reduce this case to that of integer α's. To this end let us introduce another version of R_α. Given $\Delta = [a, b]$ in $[0, 1]$ define R_α^0 on $L_2[0, b]$ by

$$(R_\alpha^0 f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s)\, ds : 0 \leqslant t < a, \\ \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} f(s)\, ds : a \leqslant t \leqslant b. \end{cases} \tag{3.5}$$

For $\alpha > 0$ this is a well-defined operator with values in $L_2[0, b]$ while for $\alpha > \frac{1}{2}$ it has even values in $C[0, b]$. Furthermore let S_α (for $\Delta = [a, b]$ as before) be defined by (2.5). Then the following multiplication formula will play an important role later on.

Lemma 3.2. *Suppose* $\alpha > \frac{1}{2}$ *and* $\beta > 0$. *Then we have*

$$S_{\alpha+\beta} = R_\alpha^0 \circ S_\beta + S_\alpha \circ R_\beta^0. \tag{3.6}$$

Here S_β *and* R_β^0 *are regarded as operators into* $L_2(\Delta)$ *and into* $L_2[0, b]$, *respectively. In particular, if* $\alpha \in \mathbb{N}$, *then*

$$S_{\alpha+\beta} = R_\alpha^0 \circ S_\beta + F_\alpha, \tag{3.7}$$

where F_α *is an operator of rank less or equal than* α.

Proof. To verify (3.7) we first observe that

$$S_\alpha = R_\alpha - R_\alpha^0 \tag{3.8}$$

and

$$R_{\alpha+\beta} = R_\alpha \circ R_\beta \quad \text{as well as } R_{\alpha+\beta}^0 = R_\alpha^0 \circ R_\beta^0. \tag{3.9}$$

Consequently, by (3.8)

$$\begin{aligned} R_{\alpha+\beta} &= R_\alpha \circ R_\beta = [R_\alpha^0 + S_\alpha] \circ [R_\beta^0 + S_\beta] \\ &= R_{\alpha+\beta}^0 + S_\alpha \circ R_\beta^0 + R_\alpha^0 \circ S_\beta + S_\alpha \circ S_\beta. \end{aligned} \tag{3.10}$$

Note that S_β maps into $L_2(\Delta)$ while S_α is defined on $L_2[0,a]$, thus $S_\alpha \circ S_\beta = 0$. Since

$$S_{\alpha+\beta} = R_{\alpha+\beta} - R^0_{\alpha+\beta},$$

from (3.10) we derive (3.6).

If $\alpha \in \mathbb{N}$, by Lemma 2.1 we get $\mathrm{rk}(S_\alpha) \leqslant \alpha$. Thus setting $F_\alpha := S_\alpha \circ R^0_\beta$ by (3.6) this immediately leads to (3.7). □

The following observation about representation (3.6) will be important later on: Since S_β maps $L_2[0,a]$ into $L_2(\Delta)$, by the definition of R^0_α (compare (3.5)) the first term in (3.6) may also be written as

$$R^0_\alpha \circ S_\beta = [R_\alpha : L_2(\Delta) \to C(\Delta)] \circ [S_\beta : L_2[0,1] \to L_2(\Delta)]. \tag{3.11}$$

Here the first operator at the right-hand side of (3.11) has to be understood as the restriction of R_α to functions in $L_2[0,1]$ having their support in Δ.

Consequently, in view of (2.4) we obtain the following result.

Corollary 3.3. *Let $\Delta_1, \dots, \Delta_m$ be as before and suppose that $\alpha \in \mathbb{N}$. Then for any $\beta > 0$ we have*

$$R_{\alpha+\beta} - R^\Delta_{\alpha+\beta} = R^\Delta_\alpha \circ \left(\sum_{j=1}^m S^j_\beta \right) + F, \tag{3.12}$$

where F is an operator with $\mathrm{rk}(F) \leqslant \alpha m$.

In view of (3.12) it is necessary to get more information about the degree of compactness of the operators S^j_β, $0 < \beta < 1$, regarded as mappings into $L_2(\Delta)$.

Lemma 3.4. *Define $S_\beta : L_2[0,a] \to L_2(\Delta)$, $\Delta = [a,b]$, by*

$$(S_\beta f)(t) := \frac{1}{\Gamma(\beta)} \int_0^a (t-s)^{\beta-1} f(s)\, ds.$$

If $0 < \beta < 1$, then there are constants $c, c_\beta > 0$ (independent of Δ) such that for all $n \geqslant 2$

$$d_n(S_\beta) \leqslant c \cdot e^{-c_\beta n^{1/2}} \cdot |\Delta|^\beta. \tag{3.13}$$

Proof. We split the proof into three steps. In a first one we investigate the operator S^∞_β mapping $L_2[1,\infty)$ into $C[0,1]$ defined by

$$(S^\infty_\beta f)(t) := \int_1^\infty [(t+s)^{\beta-1} - s^{\beta-1}] f(s)\, ds$$

and we claim that

$$d_n(S^\infty_\beta : L_2[1,\infty) \to C[0,1]) \leqslant c \cdot \frac{\Gamma(n+3-\beta)}{\Gamma(n+2)} \cdot 2^{-n}. \tag{3.14}$$

To verify this take $f \in L_2[1,\infty)$ and let $P_n(S_\beta^\infty f;t)$ be the nth Taylor polynomial of $S_\beta^\infty f$ taken at the point $t_0 = \frac{1}{2}$. Then it follows that

$$|(S_\beta^\infty f)(t) - P_n(S_\beta^\infty f;t)| \leqslant \frac{1}{2^n} \cdot \frac{(1-\beta)(2-\beta)\dots(n+2-\beta)}{(n+1)!} \int_1^\infty s^{-n-1+\beta}|f(s)|\,ds$$

$$\leqslant c \cdot \frac{\Gamma(n+3-\beta)}{\Gamma(n+2)} \cdot 2^{-n}||f||_2$$

which proves (3.14).

In a second step we fix a number $\Lambda \geqslant 1$ and define an operator $S_\beta^\Lambda : L_2[0,\Lambda] \to L_2[0,1]$ by

$$(S_\beta^\Lambda f)(t) := \int_0^\Lambda (t+s)^{\beta-1} f(s)\,ds. \tag{3.15}$$

We are going to prove that for $n \geqslant 2$

$$d_n(S_\beta^\Lambda : L_2[0,\Lambda] \to L_2[0,1]) \leqslant c \cdot e^{-c_\beta n^{1/2}} \tag{3.16}$$

with $c, c_\beta > 0$ independent of Λ. To this end write

$$S_\beta^\Lambda = S_\beta^{(1)} + S_\beta^{(2)} + F, \tag{3.17}$$

where

$$(S_\beta^{(1)} f)(t) := \int_0^1 (t+s)^{\beta-1} f(s)\,ds,$$

$$(S_\beta^{(2)} f)(t) := \int_1^\Lambda [(t+s)^{\beta-1} - s^{\beta-1}] f(s)\,ds$$

and the operator F is defined by

$$(Ff)(t) := \int_1^\Lambda s^{\beta-1} f(s)\,ds.$$

A result of Laptev (cf. [14]) asserts

$$d_n(S_\beta^{(1)} : L_2[0,1] \to L_2[0,1]) \leqslant c \cdot e^{-c_\beta n^{1/2}} \tag{3.18}$$

and (3.14) lets us conclude

$$d_n(S_\beta^{(2)} : L_2[1,\Lambda] \to L_2[0,1]) \leqslant c \cdot \frac{\Gamma(n+3-\beta)}{\Gamma(n+2)} \cdot 2^{-n} \tag{3.19}$$

with $c > 0$ independent of Λ. Of course, $\mathrm{rk}(F) = 1$, hence by (3.17)

$$d_{2n}(S_\beta^\Lambda) \leqslant d_n(S_\beta^{(1)}) + d_n(S_\beta^{(2)}),$$

i.e. by (3.18) and (3.19) we have $d_n(S_\beta^\Lambda) \leqslant c \cdot e^{-c_\beta n^{1/2}}$ as long as $n \geqslant 2$. This proves (3.16).

In a last step we verify now (3.13). Thus put $\delta := |\Delta|$. By isometric transformations (change of variables) it follows that

$$d_n(S_\beta) = \delta^\beta \cdot d_n(\tilde{S}_\beta), \tag{3.20}$$

where $\tilde{S}_\beta$ maps $L_2[0, 1/\delta]$ into $L_2[0,1]$ and

$$(\tilde{S}_\beta f)(t) := \int_0^{1/\delta} (t+s)^{\beta-1} f(s)\, ds.$$

Of course, by (3.16) (with $\Lambda = 1/\delta$) and by (3.20) we finally get (3.13) as asserted. $\square$

Corollary 3.5. *Let $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant 0$ be the singular numbers of S_β. Then,*

$$\sigma_n \leqslant c \cdot e^{-c_\beta n^{1/2}} \cdot |\Delta|^\beta$$

provided that $n \geqslant 2$. Hence

$$l_2(S_\beta) = \left(\sum_{n=2}^{\infty} \sigma_n^2 \right)^{1/2} \leqslant c \cdot |\Delta|^\beta. \tag{3.21}$$

Remark. It is not difficult to see that (3.21) remains true for $l_1(S_\beta) = l(S_\beta)$ provided that $0 < \beta < \frac{1}{2}$ while it is no longer valid for $l(S_\beta)$ when $\frac{1}{2} \leqslant \beta < 1$.

We are now in the position to extend Proposition 2.2 to fractional integration operators with arbitrary index.

Proposition 3.6. *Let $\Delta_1, \ldots, \Delta_m$ be as before intervals in $[0,1]$ with disjoint interior and with union Δ and suppose $\sup_{1 \leqslant j \leqslant m} |\Delta_j| \leqslant \delta$. Then for $\alpha \geqslant 1$ there is a natural number $\kappa = \kappa(\alpha)$ such that*

$$d_{\kappa m}(R_\alpha : L_2[0,1] \to C(\Delta)) \leqslant c \cdot m^{-1/2} \cdot \delta^{\alpha - 1/2}. \tag{3.22}$$

Proof. Given $\alpha \geqslant 1$, in view of Proposition 2.2 we may suppose that $\alpha = k + \beta$ where $k \in \mathbb{N}$ and $0 < \beta < 1$. By (3.12) we get

$$R_\alpha - R_\alpha^\Delta = R_k^\Delta \circ S + F,$$

where $S = \sum_{j=1}^m S_\beta^j$ and $\operatorname{rk}(F) \leqslant km$, consequently,

$$d_{m(2k+5)}(R_\alpha - R_\alpha^\Delta) \leqslant d_{m(k+1)}(R_k^\Delta) \cdot d_{4m}(S). \tag{3.23}$$

We estimate now both terms on the right-hand side of (3.23) separately. Because of (2.11), (2.10) and (2.9) we obtain

$$d_{m(k+1)}(R_k^\Delta) \leqslant c \cdot m^{-1/2} \cdot \delta^{k-1/2} \tag{3.24}$$

while Lemma 3.1 for $n = 2$ together with (3.21) yields

$$\sqrt{2m} \cdot d_{4m}(S) \leqslant \left(\sum_{j=1}^m l_2(S_\beta^j)^2 \right)^{1/2} \leqslant c \cdot m^{1/2} \cdot \delta^\beta. \tag{3.25}$$

Combining (3.23)–(3.25) finally gives

$$d_{m(2k+5)}(R_\alpha - R_\alpha^\Delta) \leqslant c \cdot m^{-1/2} \cdot \delta^{\alpha-1/2},$$

thus for each $l \in \mathbb{N}$ we conclude

$$d_{m(2k+5+l)}(R_\alpha) \leqslant d_{ml}(R_\alpha^\Delta) + c \cdot m^{-1/2} \cdot \delta^{\alpha-1/2} \tag{3.26}$$

and

$$d_{m(2k+5+l)}(R_\alpha^\Delta) \leqslant d_{ml}(R_\alpha) + c \cdot m^{-1/2} \cdot \delta^{\alpha-1/2}. \tag{3.27}$$

We argue now as in the proof of Proposition 2.2, i.e. we first apply (3.26) with $l = 2k+6$, then (2.11), next (3.27) with $l = 1$ and finally (1.3). Doing so it follows that

$$\begin{aligned} d_{m(4k+11)}(R_\alpha) &\leqslant d_{m(2k+6)}(R_\alpha^\Delta) + c_1 \cdot m^{-1/2} \cdot \delta^{\alpha-1/2} \\ &= d_{m(2k+6)}(R_\alpha^{\tilde{\Delta}}) + c_1 \cdot m^{-1/2} \cdot \delta^{\alpha-1/2} \\ &\leqslant d_m(\tilde{R}_\alpha) + c_2 \cdot m^{-1/2} \cdot \delta^{\alpha-1/2} \\ &\leqslant c_3 \cdot m^{-1/2} \cdot \delta^{\alpha-1/2} \end{aligned}$$

This completes the proof with $\kappa = 4\,[\alpha] + 11$. □

Remark. We do not know whether or not (3.22) remains valid for $\frac{1}{2} < \alpha < 1$. At least our methods do not apply for those α's.

Proof of Theorem 1.2. The assertion follows from Proposition 3.6 exactly in the same way as in the case $\alpha \in \mathbb{N}$ (where we used Proposition 2.2 instead).

4. Lower estimates

The aim of the present section is to prove Theorem 1.4. Again we start with the investigation of integer α's.

Lemma 4.1. *Let $K \subseteq [0,1]$ be a compact set and suppose that there are $s_1, \ldots, s_m \in K$ such that*

$$|s_i - s_j| \geqslant \delta, \quad i \neq j. \tag{4.1}$$

If $I \subseteq [0,1]$ is defined by

$$I := \bigcup_{j=1}^{m} \left[s_j - \frac{\delta}{2}, s_j + \frac{\delta}{2} \right],$$

then for $\alpha \in \mathbb{N}$ it follows that

$$d_n(R_\alpha : L_2(I) \to C(K)) \geqslant c \cdot n^{-1/2} \cdot \log\left(\frac{me}{n}\right)^{1/2} \cdot \delta^{\alpha-1/2}, \quad 1 \leqslant n \leqslant m.$$

Proof. We choose a function $\psi : \mathbb{R} \to [0, \infty)$ possessing the following properties:

(i) $\mathrm{supp}(\psi) \subset (0, 1)$,
(ii) $\psi(1/2) = 1$ and
(iii) ψ is α-times continuously differentiable.

Setting $\varphi := \psi^{(\alpha)}$, we also have $\mathrm{supp}(\varphi) \subset (0, 1)$ and, moreover, because of $\alpha \in \mathbb{N}$ it follows that $R_\alpha \varphi = \psi$. With the help of this function φ we construct now functions φ_j, $1 \leqslant j \leqslant m$, by

$$\varphi_j(s) := \varphi\left(\frac{s - s_j + \delta/2}{\delta}\right), \quad s \in \mathbb{R},$$

satisfying

$$\|\varphi_j\|_2 = \delta^{1/2} \cdot \|\varphi\|_2 \quad \text{and} \quad \mathrm{supp}(\varphi_j) \subset [s_j - \delta/2, s_j + \delta/2]. \tag{4.2}$$

Furthermore,

$$(R_\alpha \varphi_j)(t) = \delta^\alpha \, (R_\alpha \varphi)\left(\frac{t - s_j + \delta/2}{\delta}\right) = \delta^\alpha \, \psi\left(\frac{t - s_j + \delta/2}{\delta}\right)$$

leads by property (ii) of ψ to

$$(R_\alpha \varphi_j)(s_j) = \delta^\alpha, \quad 1 \leqslant j \leqslant m. \tag{4.3}$$

Next we define an operator $B : l_2^m \to L_2(I)$ by

$$B(x) := \sum_{j=1}^{m} x_j \varphi_j, \quad x = (x_1, \ldots, x_m),$$

which by (4.2) satisfies

$$\|B(x)\|_2 = \delta^{1/2} \cdot \|\varphi\|_2 \cdot \|x\|_2.$$

Another operator $\Phi : C(K) \to l_\infty^m$ is given by

$$\Phi(f) := (f(s_j))_{j=1}^m, \quad f \in C(K).$$

Of course, $\|\Phi\| \leqslant 1$ and in view of (4.3) it follows that

$$(\Phi \circ R_\alpha \circ B)(x) = \delta^\alpha \cdot x, \quad x \in l_2^m,$$

i.e. we have

$$\Phi \circ R_\alpha \circ B = \delta^\alpha \cdot i_{2,\infty}$$

where $i_{2,\infty} : l_2^m \to l_\infty^m$ denotes the canonical identity map. Consequently,

$$\delta^\alpha \cdot d_n(i_{2,\infty}) \leqslant \|B\| \cdot d_n(\Phi \circ R_\alpha : L_2(I) \to l_\infty^m) \leqslant \delta^{1/2} \cdot \|\varphi\|_2 \cdot d_n(R_\alpha : L_2(I) \to C(K))$$

which completes the proof because of a deep Theorem of Garnaev and Gluskin (cf. [10]) asserting

$$d_n(i_{2,\infty}) \geqslant c \cdot n^{-1/2} \log\left(\frac{me}{n}\right)^{1/2}, \quad 1 \leqslant n \leqslant m. \qquad \square \tag{4.4}$$

As an immediate consequence of Lemma 4.1 we get the following stronger version of Theorem 1.4 in the case of integer α's.

Proposition 4.2. *Suppose $\alpha \in \mathbb{N}$ and regard R_α as operator from $L_2[0,1]$ to $C(K)$ for a certain compact set $K \subseteq [0,1]$. Then it follows that*

$$d_m(R_\alpha : L_2[0,1] \to C(K)) \geqslant c \cdot m^{-1/2} \cdot \varepsilon_m(K)^{\alpha-1/2}.$$

Proof. Given $m \in \mathbb{N}$ we choose a δ with $\varepsilon_m(K)/2 \leqslant \delta < \varepsilon_m(K)$. Then there are $s_1, \ldots, s_m$ satisfying (4.1), hence Lemma 4.1 applies with $n = m$. Note that, of course,

$$d_m(R_\alpha : L_2(I) \to C(K)) \leqslant d_m(R_\alpha : L_2[0,1] \to C(K)).$$

This completes the proof. □

Before treating the non-integer case we need another lemma for later purposes.

Lemma 4.3. *Let $\Delta_1, \ldots, \Delta_m$ be intervals in $[0,1]$ with disjoint interior and $|\Delta_j| \leqslant \delta$, $1 \leqslant j \leqslant m$. Then for $\beta \in (0,1)$ there is a $\kappa \in \mathbb{N}$ such that*

$$d_{\kappa m}(R_\beta : L_2[0,1] \to L_2(\Delta)) \leqslant c \cdot \delta^\beta, \tag{4.5}$$

where, as before, $\Delta = \bigcup_{j=1}^m \Delta_j$.

Proof. The proof follows almost exactly as that of Proposition 3.6 and we use the same notation as there. Writing $R_\beta = S_\beta + R_\beta^\Delta$ by (3.25) it follows that $d_{4m}(S_\beta) \leqslant c \cdot \delta^\beta$. Hence, if $n \in \mathbb{N}$, we obtain the estimates

$$d_{n+4m}(R_\beta) \leqslant c \cdot \delta^\beta + d_n(R_\beta^\Delta) \tag{4.6}$$

as well as

$$d_{n+4m}(R_\beta^\Delta) \leqslant c \cdot \delta^\beta + d_n(R_\beta). \tag{4.7}$$

In contrast to the proof of Proposition 3.6 here $\tilde{R}_\beta$ is an operator from $L_2[0,|\Delta|]$ into $L_2[0,|\Delta|]$, thus by (1.3)

$$d_n(\tilde{R}_\beta) \leqslant n^{-\beta} \cdot |\Delta|^\beta. \tag{4.8}$$

Summing up, by (4.6)–(4.8) we finally get

$$\begin{aligned} d_{9m}(R_\beta) &\leqslant c \cdot \delta^\beta + d_{5m}(R_\beta^\Delta) = c \cdot \delta^\beta + d_{5m}(R_\beta^{\tilde{\Delta}}) \\ &\leqslant c' \cdot \delta^\beta + d_m(\tilde{R}_\beta) \leqslant c'' \cdot \delta^\beta. \end{aligned}$$

This proves the assertion with $\kappa = 9$. □

Now we are in position to prove Theorem 1.4.

Proof of Theorem 1.4. First observe that it remains to prove (1.9) for non-integer α's. Set

$$k := [\alpha] + 1 \quad \text{and} \quad \beta := k - \alpha. \tag{4.9}$$

Consequently, we have $0 < \beta < 1$.

For the proof we need a covering $\Delta_1, \ldots, \Delta_m$ of K as well as points $s_1, \ldots, s_M$ in K with sufficiently large distance. Let us start with the construction of the covering. For a given $m \in \mathbb{N}$ we choose intervals $\Delta_1, \ldots, \Delta_m$ with disjoint interior such that $K \subseteq \bigcup_{j=1}^m \Delta_j := \Delta$ and, moreover, $\sup_{1 \leqslant j \leqslant m} |\Delta_j| \leqslant 2 \cdot \varepsilon_m(K)$. Next we choose some well separated points in K. For a number $M > 2m$ which will be specified later on take $\delta > 0$ such that

$$\frac{\varepsilon_M(K)}{2} \leqslant \delta < \varepsilon_M(K). \tag{4.10}$$

Then there are $s_1, \ldots, s_M \in K$ for which $|s_i - s_j| \geqslant \delta$ if $i \neq j$. With these s_j's we define now intervals I_j possessing disjoint interiors by

$$I_j := \left[s_j - \frac{\delta}{2}, s_j + \frac{\delta}{2} \right], \quad 1 \leqslant j \leqslant M.$$

Let $D \subseteq \{1, \ldots, M\}$ be the set $D := \{j \leqslant M : I_j \subseteq \Delta\}$. Then it follows that $\bar{m} := \#(D)$ satisfies $\bar{m} \geqslant M - 2m$ and, moreover,

$$I := \bigcup_{j \in D} I_j \subseteq \Delta. \tag{4.11}$$

For a more precise formulation of the following arguments we denote by J_I and J_Δ the canonical embeddings from $L_2(I)$ and $L_2(\Delta)$ into $L_2[0,1]$, respectively. If k is defined by (4.9), then by Lemma 4.1 we obtain for each $n \in \mathbb{N}$ the estimate

$$\begin{aligned} d_n(R_k \circ J_\Delta : L_2(\Delta) \to C(K)) &\geqslant d_n(R_k \circ J_I : L_2(I) \to C(K)) \\ &\geqslant c \cdot n^{-1/2} \cdot \log\left(\frac{\bar{m}e}{n}\right)^{1/2} \cdot \delta^{k-1/2}. \end{aligned} \tag{4.12}$$

Next we apply the semigroup property of the Riemann–Liouville operators. Recall that $k = \alpha + \beta$. Doing so we obtain

$$[R_k \circ J_\Delta : L_2(\Delta) \to C(K)] = [R_\alpha : L_2[0,1] \to C(K)] \circ [R_\beta \circ J_\Delta : L_2(\Delta) \to L_2[0,1]]$$

and, consequently, by (4.12) for any $l \in \mathbb{N}$ with $l + m \leqslant \bar{m}$ it follows that

$$\begin{aligned} &c \cdot (l+m)^{-1/2} \cdot \log\left(\frac{\bar{m}e}{l+m}\right)^{1/2} \cdot \delta^{k-1/2} \\ &\quad \leqslant d_m(R_\alpha : L_2[0,1] \to C(K)) \cdot d_l(R_\beta \circ J_\Delta : L_2(\Delta) \to L_2[0,1]). \end{aligned} \tag{4.13}$$

We claim now that for $0 < \beta < 1$ there exists a natural number κ with

$$d_{\kappa m}(R_\beta \circ J_\Delta : L_2(\Delta) \to L_2[0,1]) \leqslant c \cdot \varepsilon_m(K)^\beta. \tag{4.14}$$

Assume for a moment that (4.14) has already been proven. Then we may precise the choice of the number M from above as $M := (\kappa + 3)m$, hence $\bar{m} \geqslant (\kappa + 1)m$, and using (4.13) with $l := \kappa m$ we derive from (4.14) and (4.10) that

$$c' \cdot \varepsilon_M(K)^{k-1/2} \cdot m^{-1/2} \leqslant d_m(R_\alpha : L_2[0,1] \to C(K)) \cdot d_{\kappa m}(R_\beta \circ J_\Delta : L_2(\Delta) \to L_2[0,1]) \leqslant c \cdot d_m(R_\alpha : L_2[0,1] \to C(K)) \cdot \varepsilon_m(K)^\beta. \tag{4.15}$$

In view of (1.8) we have

$$\varepsilon_m(K)^{k-1/2} \leqslant \rho \cdot \varepsilon_M(K)^{k-1/2}$$

for a certain $\rho \geqslant 1$ depending on λ in (1.8) and on α. Hence (4.15) leads to the desired estimate

$$d_m(R_\alpha : L_2[0,1] \to C(K)) \geqslant c \cdot m^{-1/2} \cdot \varepsilon_m(K)^{k-\beta-1/2} = c \cdot m^{-1/2} \cdot \varepsilon_m(K)^{\alpha-1/2}.$$

Consequently, to complete the proof it remains to verify (4.14). First note that for any $n \in \mathbb{N}$

$$d_n(R_\beta \circ J_\Delta : L_2(\Delta) \to L_2[0,1]) = d_n(J_\Delta^* R_\beta^* : L_2[0,1] \to L_2(\Delta)), \tag{4.16}$$

where the dual operator $J_\Delta^* R_\beta^*$ acts as

$$(J_\Delta^* R_\beta^* f)(s) = \frac{1}{\Gamma(\beta)} \int_s^1 (t-s)^{\beta-1} f(t)\, dt, \quad s \in \Delta.$$

By an easy isometric transformation $J_\Delta^* R_\beta^*$ may be transformed to $\bar{R}_\beta : L_2[0,1] \to L_2(\bar{\Delta})$ with

$$(\bar{R}_\beta f)(u) := \frac{1}{\Gamma(\beta)} \int_0^u (u-v)^{\beta-1} f(v)\, dv, \quad u \in \bar{\Delta}.$$

Here the set $\bar{\Delta}$ is defined by $\bar{\Delta} := \bar{\Delta}_1 \cup \cdots \cup \bar{\Delta}_m$ with $\bar{\Delta}_j := \{1-u : u \in \Delta_j\}$ for $1 \leqslant j \leqslant m$. Observe that $|\bar{\Delta}_j| = |\Delta_j| \leqslant 2 \cdot \varepsilon_m(K)$, hence we are in the situation of Lemma 4.3 and this leads to

$$d_{\kappa m}(\bar{R}_\beta) \leqslant c \cdot \varepsilon_m(K)^\beta, \tag{4.17}$$

where, for example, κ may be chosen as $\kappa = 9$. In view of (4.16) and using $d_n(J_\Delta^* R_\beta^*) = d_n(\bar{R}_\beta)$ by (4.17) we get the desired estimate (4.14). This completes the proof of Theorem 1.4. □

Questions. 1. We do not know whether or not the regularity condition (1.8) is indeed necessary for the lower estimate in the non-integer case. Note that for integer α's Theorem 1.4 holds without any extra assumption about the behavior of $\varepsilon_m(K)$.

2. We believe that Theorem 1.4 is also valid for the entropy numbers of R_α, yet could not verify this because we do not know whether or not (4.5) is true for the entropy numbers as well. But observe that (1.9) holds for integer α's. Indeed, using instead of (4.4) an estimate for the entropy numbers of the embedding $i_{2,\infty}$ due to Schütt (cf. [28]) by the same arguments as before

$$e_m(R_\alpha : L_2[0,1] \to C(K)) \geqslant c \cdot m^{-1/2} \cdot \varepsilon_m(K)^{\alpha-1/2}$$

whenever $\alpha \in \mathbb{N}$.

5. Metric entropy of fractal sets

In view of Theorems 1.2 and 1.4 it is important to find precise upper and lower estimates for $\varepsilon_m(K)$ where $K \subset [0,1]$ is a compact set with $|K| = 0$. Before stating a general representation theorem for those sets let us introduce the following class of functions on $[0,1]$:

$$\mathcal{A} := \left\{ A : [0,1] \to [0,1] : A(t) \right.$$
$$\left. = \sum_{\tau_k \leqslant t} \alpha_k \text{ for certain } \tau_k \in (0,1), \alpha_k \geqslant 0, \sum_{k=1}^{\infty} \alpha_k = 1 \right\}. \tag{5.1}$$

Note that $\mathcal{A}$ is exactly the set of distribution functions of discrete probability measures on $(0,1)$.

Proposition 5.1. *Let $K \subset [0,1]$ be compact with*

$$0 \in K, \quad 1 \in K \quad \text{and} \quad |K| = 0. \tag{5.2}$$

Then there is a function $A \in \mathcal{A}$ such that

$$K = \overline{\{A(t) \colon 0 \leqslant t \leqslant 1\}}. \tag{5.3}$$

Proof. We may represent the complement K^c of K (taken in $[0,1]$) in the form

$$K^c = \bigcup_{k=1}^{\infty} G_k \tag{5.4}$$

with open, disjoint intervals $G_1, G_2, \ldots$. Setting $\alpha_k := |G_k|$, $k = 1, 2, \ldots$, in view of $|K| = 0$ we obtain

$$\sum_{k=1}^{\infty} \alpha_k = 1. \tag{5.5}$$

By induction we choose now real numbers $\tau_k \in (0,1)$ such that $\tau_k < \tau_l$ for certain $1 \leqslant k \neq l < \infty$ iff the interval G_k is on the left hand side of G_l. With these α_k's and τ_k's we define a function A by

$$A(t) := \sum_{\tau_k \leqslant t} \alpha_k, \quad 0 \leqslant t \leqslant 1.$$

Because of (5.5) it follows that $A \in \mathcal{A}$ and, moreover, by the construction of the τ_k's it holds

$$A(\tau_k - 0) = \sum_{\tau_l < \tau_k} |G_l| \quad \text{and} \quad A(\tau_k) = \sum_{\tau_l \leqslant \tau_k} |G_l|,$$

thus $G_k = (A(\tau_k - 0), A(\tau_k))$ for $k = 1, 2, \ldots$. Take now $x \in [0,1]$. This element does not belong to the closure of the range of A iff there is a number $k \geqslant 1$

with $A(\tau_k - 0) < x < A(\tau_k)$, hence

$$[0,1] \setminus \overline{\{A(t) : 0 \leqslant t \leqslant 1\}} = \bigcup_{k=1}^{\infty} (A(\tau_k - 0), A(\tau_k)) = K^c$$

completing the proof. □

Remarks.

(1) Of course, in representation (5.3) the function $A \in \mathcal{A}$ may always be chosen such that the weights α_k tend to zero monotonely.
(2) It is not difficult to see that, conversely, every K represented as in (5.3) (with a suitable function $A \in \mathcal{A}$) satisfies (5.2).

Proposition 5.2. *Suppose a compact set $K \subset [0,1]$ admits representation* (5.3) *with a function $A \in \mathcal{A}$ for which $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant 0$. Then it follows that*

$$\varepsilon_{2m+1}(K) \leqslant \frac{1}{m} \sum_{k=m+1}^{\infty} \alpha_k \tag{5.6}$$

while

$$\varepsilon_m(K) \geqslant \frac{\alpha_m}{2}. \tag{5.7}$$

Proof. Let us first verify (5.6). Suppose the complement of K in $[0,1]$ is represented as in (5.4), i.e.

$$K^c = \bigcup_{k=1}^{\infty} G_k \quad \text{with} \quad |G_k| = \alpha_k.$$

Fix $m \in \mathbb{N}$ now and let $C_0, \ldots, C_m$ be the gaps between $G_1, \ldots, G_m$, i.e. $C_0, \ldots, C_m$ are closed disjoint intervals (also disjoint to each G_k, $1 \leqslant k \leqslant m$) with

$$\bigcup_{j=0}^{m} C_j \cup \bigcup_{k=1}^{m} G_j = [0,1].$$

Then we obtain

$$K \subseteq \bigcup_{j=0}^{m} C_j \tag{5.8}$$

as well as

$$\sum_{j=0}^{m} |C_j| = 1 - \sum_{k=1}^{m} |G_k| = \sum_{k=m+1}^{\infty} \alpha_k. \tag{5.9}$$

Given a number $\delta > 0$ each set C_j may be covered by at most $\delta^{-1} \cdot |C_j| + 1$ open intervals of length less than δ. Consequently, by (5.8) the set K admits a

δ-cover by at most

$$\delta^{-1} \cdot \sum_{j=0}^{m} |C_j| + m + 1 \tag{5.10}$$

open intervals. Applying (5.10) with

$$\delta = \delta_m = \frac{1}{m} \sum_{k=m+1}^{\infty} \alpha_k,$$

in view of (5.9) the set K may be covered by $2m+1$ intervals of length less than δ_m. In other words,

$$\varepsilon_{2m+1}(K) \leqslant \delta_m = \frac{1}{m} \sum_{k=m+1}^{\infty} \alpha_k$$

as asserted.

Estimate (5.7) is even easier to prove. Indeed, fix again $m \in \mathbb{N}$ and define elements $s_k \in [0,1]$, $1 \leqslant k \leqslant m$, by $s_k := A(\tau_k)$. By the construction we have $s_k \in K$, and for $k \neq l$ there is either an interval of length α_k or of length α_l in-between s_k and s_l. Since the α_k's are decreasing this implies

$$|s_k - s_l| \geqslant \min\{\alpha_k, \alpha_l\} \geqslant \alpha_m.$$

Thus there exist m elements in K with mutually distance of at least α_m yielding $\varepsilon_m(K) \geqslant \alpha_m/2$ as asserted.

Corollary 5.3. *Suppose that the weights α_k of $A \in \mathcal{A}$ satisfy $\alpha_k \approx k^{-\theta} \cdot (\log k)^{\beta}$ for some $\theta > 1$ and $\beta \in \mathbb{R}$ and let $K \subset [0,1]$ be generated by A as in* (5.3). *Then independent of the choice of the τ_k's we have*

$$\varepsilon_m(K) \approx m^{-\theta} \cdot (\log m)^{\beta}.$$

Remark. If either the α_k's tend to zero very rapidly, e.g. $\alpha_k = 2^k$, or very slowly, e.g. $\alpha_k \approx k^{-1} \cdot (\log k)^{-\beta}$ for some $\beta > 1$, then Proposition 5.2 does not lead to sharp estimates for $\varepsilon_m(K)$. Here the τ_k's (more precisely, the way how the τ_k's are ordered) are important for the degree of compactness of K.

Example. Let us treat as an example the classical Cantor set $\mathbf{C}$ in $[0,1]$. As can be seen easily, this set may be generated by a function A where the decreasing weights α_k satisfy $\alpha_k \approx k^{-\log 3/\log 2}$. Consequently, it follows that

$$d_m(R_\alpha : L_2[0,1] \to C(\mathbf{C})) \approx e_m(R_\alpha : L_2[0,1] \to C(\mathbf{C})) \approx m^{-1/2-\theta(\alpha-1/2)},$$

where $\theta = \log 3/\log 2$.

Proposition 5.2 allows us to reformulate Theorems 1.2 and 1.4. Given a function $A \in \mathcal{A}$ (we always suppose now that the weights α_k are in decreasing order) and $\alpha > \frac{1}{2}$

we may define an operator $R_\alpha^A : L_2[0,1] \to C[0,1]$ as follows.

$$(R_\alpha^A f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^{A(t)} (A(t)-s)^{\alpha-1} f(s)\,ds, \quad 0 \leqslant t \leqslant 1. \tag{5.11}$$

Proposition 5.4. *For $A \in \mathcal{A}$ with (decreasing) weights α_k let R_α^A be defined by* (5.11). *Then the following are valid.*

(1) *For certain $\kappa \in \mathbb{N}$ and $c > 0$ it follows that*

$$d_{\kappa m}(R_\alpha^A : L_2[0,1] \to C[0,1]) \leqslant c \cdot m^{-\alpha} \cdot \left(\sum_{k=m+1}^{\infty} \alpha_k \right)^{\alpha - 1/2}. \tag{5.12}$$

(2) *If $\alpha_k \approx k^{-\theta} (\log k)^\beta$ for some $\theta > 1$ and $\beta \in \mathbb{R}$, then this implies*

$$\begin{aligned} d_m(R_\alpha^A : L_2[0,1] \to C[0,1]) &\approx e_m(R_\alpha^A : L_2[0,1] \to C[0,1]) \\ &\approx m^{-1/2-\theta(\alpha-1/2)} \cdot (\log m)^{\beta(\alpha-1/2)}. \end{aligned} \tag{5.13}$$

Remark. Suppose that the weights α_k of the function A are not necessarily normalized, i.e. it holds $\sum_{k=1}^\infty \alpha_k = d$ for some $d > 0$. In this case the operator R_α^A maps $L_2[0,d]$ into $C[0,1]$. Yet by the scaling properties of R_α estimates (5.12) as well as (5.13) remain valid in this more general situation (without any extra factor depending on d).

6. Probabilistic applications

Given a Hilbert space $\mathcal{H}$ and an operator $S : \mathcal{H} \to C(K)$ for a certain compact metric space K such that

$$X_S := \sum_{j=1}^{\infty} \xi_j S f_j \tag{6.1}$$

converges a.s. (in $C(K)$) for some (each) ONB $(f_j)_{j \geqslant 1}$ in $\mathcal{H}$ (here as in (3.1) the ξ_j's are i.i.d. standard normal) we may regard X_S as stochastic process indexed by K. More precisely, we set

$$X_S(t) := \sum_{j=1}^{\infty} \xi_j (S f_j)(t), \quad t \in K. \tag{6.2}$$

Note that $X_S = (X_S(t))_{t \in K}$ is then a centered Gaussian process possessing a.s. continuous paths. For example, if

$$\sum_{m=1}^{\infty} m^{-1/2} e_m(S) < \infty \tag{6.3}$$

by Dudley's theorem (cf. [8]) combined with a result in [32] the sum in (6.1) converges a.s., hence the process X_S over K is well-defined with a.s. continuous paths.

A higher degree of compactness of the operator S leads to better small deviation estimates for the process X_S. More precisely, the following was proved in [12,17].

Proposition 6.1. *Given $\gamma > 0$ and $\beta \in \mathbb{R}$ the following are equivalent.*

(i) $e_m(S) \preccurlyeq m^{-1/2-\gamma}(\log m)^{\beta}$,

(ii) $-\log \mathbb{P}\left(\sup_{t\in K} |X_S(t)| \leqslant \varepsilon\right) \preccurlyeq \varepsilon^{-1/\gamma} \log(1/\varepsilon)^{\beta/\gamma}$.

Moreover, the above equivalence remains valid for $\approx$ instead of $\preccurlyeq$ in (i) *and* (ii), *respectively.*

Here we have used the following notation: Given two functions f and g on $(0, \infty)$, then $f(\varepsilon) \preccurlyeq g(\varepsilon)$ means that there is a constant $c > 0$ such that $f(\varepsilon) \leqslant c \cdot g(\varepsilon)$ for small $\varepsilon > 0$. We write $f(\varepsilon) \approx g(\varepsilon)$ provided that $f(\varepsilon) \preccurlyeq g(\varepsilon)$ holds together with $g(\varepsilon) \preccurlyeq f(\varepsilon)$.

Let $H > 0$ be given and let K be a compact subset of $[0, 1]$. Then we regard the Riemann–Liouville operator $R_{H+1/2}$ as before as operator from $L_2[0, 1]$ into $C(K)$. In view of (1.3) for all $H > 0$ this operator satisfies (6.3), hence for any fixed ONB $(f_j)_{j\geqslant 1}$ in $L_2[0, 1]$ the process

$$W_H(t) := \sum_{j=1}^{\infty} \xi_j \, (R_{H+1/2} f_j)(t)$$
$$= \frac{1}{\Gamma(H + 1/2)} \sum_{j=1}^{\infty} \xi_j \int_0^t (t - s)^{H-1/2} f_j(s)\, ds, \quad t \in K,$$

is a well-defined centered Gaussian process with a.s. continuous paths over K. The process $W_H = (W_H(t))_{t\in K}$ is usually called (cf. [18]) Riemann–Liouville process with Hurst index H. For $H = \frac{1}{2}$ the process W_H is the Wiener process while for $H = k + \frac{1}{2}$ with $k \in \mathbb{N}$ we get the k-times (pathwise) integrated Wiener process.

A first application of Proposition 6.1 leads to the following result.

Theorem 6.2. *Suppose the compact set $K \subseteq [0, 1]$ satisfies $\varepsilon_m(K) \preccurlyeq m^{-\theta} (\log m)^{\beta}$ for some $\theta \geqslant 1$ and $\beta \in \mathbb{R}$ (again we necessarily have $\beta \leqslant 0$ for $\theta = 1$). Then this implies*

$$-\log \mathbb{P}\left(\sup_{t\in K} |W_H(t)| \leqslant \varepsilon\right) \preccurlyeq \varepsilon^{-1/(\theta H)} \cdot \log(1/\varepsilon)^{\beta/\theta}. \tag{6.4}$$

Moreover, if even $\varepsilon_m(K) \approx m^{-\theta} (\log m)^{\beta}$, then

$$-\log \mathbb{P}\left(\sup_{t\in K} |W_H(t)| \leqslant \varepsilon\right) \approx \varepsilon^{-1/(\theta H)} \cdot \log(1/\varepsilon)^{\beta/\theta}. \tag{6.5}$$

Proof. Suppose first $\varepsilon_m(K) \preccurlyeq m^{-\theta} (\log m)^{\beta}$. Then by Theorem 1.3 this implies

$$e_m(R_{H+1/2} : L_2[0,1] \to C(K)) \preccurlyeq m^{-1/2-\theta H} \cdot (\log m)^{\beta H}.$$

An application of Proposition 6.1 with $\gamma = \theta H$ and with βH easily gives (6.4) as asserted.

If even $\varepsilon_m(K) \approx m^{-\theta} (\log m)^{\beta}$, this time we may use Corollary 1.5 and obtain

$$e_m(R_{H+1/2} : L_2[0,1] \to C(K)) \approx m^{-1/2-\theta H} \cdot (\log m)^{\beta H}.$$

Another application of Proposition 6.1 (this time for $\approx$) completes the proof of (6.5). □

Example. In the special case $\mathbf{C}$ of the Cantor set it follows that

$$-\log \mathbb{P}\left(\sup_{t \in \mathbf{C}} |W_H(t)| \leqslant \varepsilon \right) \approx \varepsilon^{-\log 2/(H \log 3)}.$$

In particular, for the k-times integrated Wiener process over $\mathbf{C}$ this implies that the order of its small ball behavior (in the log-level) is $\varepsilon^{-2 \log 2/((2k+1)\log 3)}$.

In the case $0 < H < 1$ the process W_H is tightly related with the fractional Brownian motion B_H of Hurst index H. Recall that B_H is a centered Gaussian process indexed by $[0, \infty)$ with a.s. continuous paths satisfying

$$\mathbb{E} B_H(t) B_H(s) = \tfrac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad 0 \leqslant t, s < \infty.$$

The following concrete representation of B_H over $[0,1]$ turns out to be very useful. Let the Hilbert space $\mathcal{H}$ be given by $\mathcal{H} := L_2[0,1] \oplus L_2[0,\infty)$ and define $S_H : \mathcal{H} \to C[0,1]$ by

$$S_H(f \oplus g) := c_H(R_{H+1/2} f + Q_H g).$$

Here

$$c_H := \Gamma(H+1/2)\left((2H)^{-1} + \int_0^\infty ((1+s)^{H-1/2} - s^{H-1/2})^2 \, ds \right)^{-1/2}$$

and the operator $Q_H : L_2[0,\infty) \to C[0,1]$ is defined by

$$(Q_H f)(t) = \frac{1}{\Gamma(H+1/2)} \int_0^\infty [(t+s)^{H-1/2} - s^{H-1/2}] f(s) \, ds. \tag{6.6}$$

As shown in [20] (cf. also [27]) the operator S_H generates the fractional Brownian motion B_H on $[0,1]$ as stated in (6.2), i.e. we have

$$B_H(t) := \sum_{j=1}^{\infty} \xi_j (S_H f_j)(t), \quad t \in [0,1].$$

Of course, regarding S_H as operator from $\mathcal{H}$ into $C(K)$ for some compact subset $K \subseteq [0,1]$, this operator generates $(B_H(t))_{t \in K}$ in the same way. In particular, Proposition 6.1 applies and relates the behavior of $e_n(S_H : \mathcal{H} \to C(K))$ with the small ball behavior of B_H over K.

Consequently, we get the following version of Theorem 6.2 for the fractional Brownian motion.

Theorem 6.3. *Suppose the compact set* $K \subseteq [0,1]$ *satisfies* $\varepsilon_m(K) \preccurlyeq m^{-\theta} (\log m)^{\beta}$ *for some* $\theta \geqslant 1$ *and* $\beta \in \mathbb{R}$. *Then for* $0 < H < 1$ *this implies*

$$-\log \mathbb{P}\left(\sup_{t \in K} |B_H(t)| \leqslant \varepsilon \right) \preccurlyeq \varepsilon^{-1/(\theta H)} \cdot \log(1/\varepsilon)^{\beta/\theta}. \tag{6.7}$$

Moreover, if even $\varepsilon_m(K) \approx m^{-\theta} (\log m)^{\beta}$, *then*

$$-\log \mathbb{P}\left(\sup_{t \in K} |B_H(t)| \leqslant \varepsilon \right) \approx \varepsilon^{-1/(\theta H)} \cdot \log(1/\varepsilon)^{\beta/\theta}. \tag{6.8}$$

Proof. As shown in [1] the operator Q_H defined in (6.6) satisfies

$$e_m(Q_H\colon\ L_2[0,\infty) \to C[0,1]) \preccurlyeq 2^{-c\, m^{1/3}} \tag{6.9}$$

with some $c > 0$ only depending on H. Of course, then also

$$e_m(Q_H\colon\ L_2[0,\infty) \to C(K)) \preccurlyeq 2^{-c\, m^{1/3}} \tag{6.10}$$

for any compact subset $K \subseteq [0,1]$.

Suppose now $\varepsilon_m(K) \preccurlyeq m^{-\theta} (\log m)^{\beta}$. Then by Theorem 1.3 this implies

$$e_m(R_{H+1/2} : L_2[0,1] \to C(K)) \preccurlyeq m^{-1/2-\theta H} \cdot (\log m)^{\beta H}.$$

Thus by

$$\begin{aligned} e_{2m-1}(S_H : \mathcal{H} \to C(K)) &\leqslant e_m(c_H\, R_{H+1/2} : L_2[0,1] \to C(K)) \\ &\quad + e_m(c_H\, Q_H : L_2[0,\infty) \to C(K)) \end{aligned}$$

from (6.10) we derive

$$e_m(S_H\colon \mathcal{H} \to C(K)) \preccurlyeq m^{-1/2-\theta H} \cdot (\log m)^{\beta H}$$

as well and the proof of (6.7) may now completed as before by an application of Proposition 6.1.

Assertion (6.8) follows by similar arguments and thus we omit the proof.

Remark. If $K = [0,1]$, thus $\theta = 1$ and $\beta = 0$, then (6.8) was first proved in [21,29]. Later on this was sharpened in [16].

Before we state another probabilistic application let us recall some facts about stable subordinators. Let $\Gamma_1 < \Gamma_2 < \cdots$ be the arrival times of a Poisson process with intensity 1 and let $\tau_1, \tau_2, \ldots$ be independent, uniformly distributed on $[0,1]$. Assume that $(\Gamma_j)_{j \geqslant 1}$ and $(\tau_j)_{j \geqslant 1}$ are independent. For some $p \in (0,1)$ define the random function A on $[0,1]$ via

$$A(t) := \sum_{\tau_k \leqslant t} \Gamma_k^{-1/p}, \quad 0 \leqslant t \leqslant 1.$$

Then A is a Lévy process over $[0,1]$, non-decreasing and p-stable (usually called p-stable subordinator, cf. [2] for more information). Let now B_H be a fractional Brownian motion of Hurst index $H \in (0,1)$ over $[0,\infty)$, independent of the p-stable subordinator A and define X_H as

$$X_H(t) := B_H(A(t)), \quad 0 \leqslant t \leqslant 1.$$

Note that for $H = 1/2$ the stochastic process $X_{1/2}$ is the so-called $2p$-stable Lévy motion. The small ball behavior of X_H (in the usual way and also conditionally, i.e. for a fixed path of A) may be derived from results for general subordinators in [19] (if $H = \frac{1}{2}$ cf. [31] for the non-conditional case). Not covered by the results in [19] is the process

$$Y_H(t) := W_H(A(t)), \quad 0 \leqslant t \leqslant 1,$$

with $H > 1$. The deeper reason is that Talagrand's small ball result (cf. [15, p. 257]), a basic ingredient in [19], does no longer apply for those H's.

For a precise formulation of the next result let us suppose that W_H is modelled over $(\Omega, \mathbb{P})$ while A is defined on $(\Omega', \mathbb{P}')$.

Proposition 6.4. *For* $0 < p < 1$ *let* A *be a* p*-stable subordinator independent of* W_H, $H > 0$. *Then for almost all* $\omega' \in \Omega'$ *we have*

$$-\log \mathbb{P}\left(\sup_{0 \leqslant t \leqslant 1} |W_H(A(t,\omega'))| \leqslant \varepsilon \right) \approx \varepsilon^{-p/H}. \tag{6.11}$$

Proof. By the Strong Law of Large Numbers it follows that $\lim_{j\to\infty} \Gamma_j / j = 1$ a.s. Hence, almost all weights α_k of the subordinator A behave like $k^{-1/p}$ and, consequently, by Proposition 5.4 and the Remark following it, for a.s. all $\omega' \in \Omega'$ the entropy numbers of $R_{H+1/2}^{A(\cdot,\omega')}$ behave like $m^{-1/2-H/p}$. Thus the assertion follows from Proposition 6.1.

Remark. Applying Fatou's Lemma to (6.11) leads to a one-sided estimate for the usual (non-conditional) small ball behavior of Y_H. More precisely, we then get

$$-\log\, (\mathbb{P} \times \mathbb{P}') \left\{ \sup_{0 \leqslant t \leqslant 1} |W_H(A(t))| \leqslant \varepsilon \right\} \preccurlyeq \varepsilon^{-p/H}.$$

The corresponding lower estimate will be treated (in a more general context) in a forthcoming paper.

Acknowledgments

The author is very grateful to Zhan Shi from Université Paris VI for many very fruitful discussions about probabilistic problems which finally initiated the investigation of the presented topic. Without his remarks and ideas this paper never would have been written. Furthermore we thank Eduard Belinsky for careful reading of the first part of the manuscript and providing several suggestions. Finally we are grateful to Martina Zähle for her helpful remarks concerning the structure of "small" compact sets in $[0, 1]$.

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